

Maximum principle for optimal control of forward-backward doubly stochastic differential equations with jumps

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Abstract

In this paper we consider the maximum principle of optimal control for a stochastic control problem. This problem is governed by a system of fully coupled multi-dimensional forward-backward doubly stochastic differential equation with Poisson jumps. Moreover, all coefficients appearing in this system are allowed to depend on the control variable.

We derive, in particular, sufficient conditions for optimality for this stochastic optimal control problem.

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1 Introduction

Forward-backward stochastic differential equations (FBSDEs in short) were first studied by Antonelli in [1], and since then they are encountered in stochastic optimal control problem and mathematical finance. For example, Xu in [17] studied a non-coupled continuous forward-backward stochastic control system. Then Wu, [15], studied extensively the maximum principle for optimal control problem of fully coupled continuous forward-backward stochastic system. Peng and Wu, [7], considered fully coupled continuous forward-backward stochastic differential equations with random coefficients and applications to optimal control. A method of continuation is developed there. Shi and Wu in [9] studied

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the maximum principle for fully coupled continuous forward-backward stochastic system, but the forward diffusion does not contain the control variable.

Fully coupled FBSDEs with respect to Brownian motion and Poisson process were considered by Wu in [16] and Yin and Situ in [12]. Such equations have been shown to be very useful for example in studying linear quadratic optimal control problems of random jumps, and also to handle nonzero-sum differential games with random jumps. The work of Wu and Wang in [14] is useful in this respect. In [4] the authors investigated stochastic maximum principle for non-coupled one-dimensional FBSDEs with jumps. Meng, [2], considered an optimal control problem of fully coupled forward-backward stochastic systems with Poisson jumps under partial information. More generally, Shi in [8] provided recently necessary conditions for optimal control of fully coupled FBSDEs with random jumps.

Backward doubly stochastic differential equations were first introduced by Pardoux and Peng in [5]. They gave a probabilistic representation of quasi linear stochastic partial differential equations. In 2003, Peng and Shi [6] introduced fully coupled forward-backward doubly stochastic differential equations (FBDSDEs in short). Such equations are generalizations of stochastic Hamilton systems. Existence and uniqueness of the solutions to (continuous) FBDSDEs with arbitrarily fixed time duration and under some monotone assumptions are established. Then the authors provided also a probabilistic interpretation for the solutions of a class of quasilinear SPDEs. In this respect we refer the reader to [11] for an application of fully coupled FBDSDEs to provide a probabilistic formula for the solution of a quasilinear stochastic partial differential-integral equation (SPDIE in short). On the other hand, another application can be found in [10], where the authors studied the maximum principle to find existence conditions for optimality for a stochastic control problem governed by a continuous forward-backward doubly stochastic system in dimension one. These are some examples to show the importance of studying FBDSDEs.

In the present work we shall consider a discontinuous situation by adding also a random jump to the forward and backward equations, and study, in particular, a stochastic control problem where the system is governed by a nonlinear fully coupled multi-dimensional FBDSDE with jumps as in the following system. More precisely, we shall establish sufficient conditions for optimality in the form of a stochastic maximum principle for this kind of systems.

Our system under study is the following:

$$\begin{cases} dy_t = b(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dt + \sigma(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dW_t \\ \quad + \int_{\Theta} \varphi(t, y_t, Y_t, z_t, Z_t, k_t, v_t, \rho) \tilde{N}(d\rho, dt) - z_t d\overleftarrow{B}_t, \\ dY_t = -f(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dt - g(t, y_t, Y_t, z_t, Z_t, k_t, v_t) d\overleftarrow{B}_t \\ \quad + Z_t dW_t + \int_{\Theta} k_t(\rho) \tilde{N}(d\rho, dt), \\ y_0 = x_0, Y_T = h(y_T), \end{cases} \quad (1.1)$$

where b, σ, φ, f, g and h are given mappings, $(W_t)_{t \geq 0}$ and $(B_t)_{t \geq 0}$ are independent Brownian motions taking their values respectively in \mathbb{R}^n and in \mathbb{R}^m , while v represents a control process and $\tilde{N}(d\rho, dt)$ is the compensated Poisson random measure associated with a Poisson point process η . Here T is a fixed positive real number.

We shall be interested in minimizing the cost functional

$$J(v) = \mathbb{E} \left[\int_0^T \ell(t, y_t, Y_t, z_t, Z_t, k_t, v_t) dt + \beta(y_T) + \gamma(Y_0) \right], \quad (1.2)$$

over the set of all admissible controls (to be described in Section 2 below).

We close this section by pointing out that such fully coupled FBDSDEP in order to provide a probabilistic formula for the solution of a quasilinear SPDIE, as will be seen in Example 3.3 in Section 3.

The paper is organized as follows. In Section 2, we formulate the problem and give various assumptions used throughout the paper. In Section 3 we introduce the adjoint equation of (1.1), state our main theorem and give an example to illustrate this theorem. Section 4 is devoted to proving the main result.

2 Formulation of the problem and assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(W_t)_{t \in [0, T]}$ and $(B_t)_{t \in [0, T]}$ be two Brownian motions taking their values in \mathbb{R}^d and \mathbb{R}^l respectively. Let η be a Poisson point process taking its values in a measurable space $(\Theta, \mathcal{B}(\Theta))$. We denote by $\nu(d\rho)$ the characteristic measure of η which is assumed to be a σ -finite measure on $(\Theta, \mathcal{B}(\Theta))$, by $N(d\rho, dt)$ the Poisson counting measure (jump measure) induced by η with compensator $\nu(d\rho)dt$, and by

$$\tilde{N}(d\rho, dt) = N(d\rho, dt) - \nu(d\rho)dt,$$

the compensation of the jump measure $N(\cdot, \cdot)$ of η . Hence $\nu(O) = \mathbb{E}[N(O, 1)]$ for $O \in \mathcal{B}(\Theta)$. We assume that these three processes W, B and η are mutually independent.

Let \mathcal{N} denote the class of \mathbb{P} -null sets of \mathcal{F} . For each $t \in [0, T]$, we define $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\eta$, where for any process $\{\pi_t\}$, we set

$$\mathcal{F}_{s,t}^\pi = \sigma(\pi_r - \pi_s; s \leq r \leq t) \vee \mathcal{N}, \mathcal{F}_t^\pi = \mathcal{F}_{0,t}^\pi.$$

For a Euclidean space E , let $\mathcal{M}^2(0, T; E)$ denote the set of jointly measurable, $\{\mathcal{F}_t\}$ -adapted processes $\{X_t, t \in [0, T]\}$ with values in E and such that

$$\mathbb{E}\left[\int_0^T |X_t|_E^2 dt\right] < \infty.$$

Similarly, denote by $\mathcal{S}^2(0, T; E)$ to the set of càdlàg, $\{\mathcal{F}_t\}$ -adapted processes $\{\mathcal{X}_t, t \in [0, T]\}$ with values in E , and satisfy:

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\mathcal{X}_t|_E^2\right] < \infty.$$

Let $L_\nu^2(\mathbb{R}^m)$ be the set of $\mathcal{B}(\Theta)$ -measurable mapping k with values in \mathbb{R}^m such that

$$|||k||| := \left[\int_\Theta |k(\rho)|_{\mathbb{R}^m}^2 \nu(d\rho)\right]^{\frac{1}{2}} < \infty.$$

Denote by $\mathcal{N}_\eta^2(0, T; \mathbb{R}^m)$ to the set of $\{\mathcal{F}_t\}$ -adapted processes $\{K_t, t \in [0, T]\}$ that take their values in $L_\nu^2(\mathbb{R}^m)$ and satisfy

$$\mathbb{E}\left[\int_0^T \int_\Theta |K_t(\rho)|_{\mathbb{R}^m}^2 \nu(d\rho) dt\right] < \infty.$$

Finally, we consider

$$\begin{aligned} \mathbb{M}^2 := & \mathcal{S}^2(0, T; \mathbb{R}^n) \times \mathcal{S}^2(0, T; \mathbb{R}^m) \times \mathcal{M}^2(0, T; \mathbb{R}^{n \times l}) \\ & \times \mathcal{M}^2(0, T; \mathbb{R}^{m \times d}) \times \mathcal{N}_\eta^2(0, T; \mathbb{R}^m). \end{aligned}$$

Then \mathbb{M}^2 is a Banach space with respect to the norm $\|\cdot\|_{\mathbb{M}^2}$ given by

$$\begin{aligned} & \|\zeta\|_{\mathbb{M}^2}^2 \\ := & \mathbb{E}\left[\sup_{0 \leq t \leq T} |y_t|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T \|z_t\|^2 dt + \int_0^T \|Z_t\|^2 dt + \int_0^T |||k_t|||^2 dt\right], \end{aligned}$$

for $\zeta. = (y., Y., z., Z., k.)$.

Let U be a non-empty subset of \mathbb{R}^n . We say that $v. : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is *admissible* if $v. \in \mathcal{M}^2(0, T; \mathbb{R}^n)$ and $v_t \in U$ a.e., \mathbb{P} -a.s. The set of admissible

controls will be denoted by \mathcal{U}_{ad} . Consider the following controlled fully coupled FBDSDE with jumps:

$$\begin{cases} dy_t = b(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dt + \sigma(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dW_t \\ \quad + \int_{\Theta} \varphi(t, y_t, Y_t, z_t, Z_t, k_t, v_t, \rho) \tilde{N}(d\rho, dt) - z_t \overleftarrow{dB}_t, \\ dY_t = -f(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dt - g(t, y_t, Y_t, z_t, Z_t, k_t, v_t) \overleftarrow{dB}_t \\ \quad + Z_t dW_t + \int_{\Theta} k_t(\rho) \tilde{N}(d\rho, dt), \\ y_0 = x_0, Y_T = h(y_T), \end{cases} \quad (2.1)$$

where the mappings

$$\begin{aligned} b &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R}^{m \times d} \times L^2_{\nu}(\mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \\ \sigma &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R} \times L^2_{\nu}(\mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}, \\ \varphi &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R}^{m \times d} \times L^2_{\nu}(\mathbb{R}^m) \times \mathbb{R}^n \times \Theta \rightarrow \mathbb{R}^n, \\ f &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R}^{m \times d} \times L^2_{\nu}(\mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \\ g &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R} \times L^2_{\nu}(\mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times l}, \\ h &: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \end{aligned}$$

are measurable (further properties to be introduced later in this section) and $v \in \mathcal{U}_{ad}$. Given a full-rank $m \times n$ matrix R of real indices, we assume that h is defined, for $(\omega, x) \in \Omega \times \mathbb{R}^n$, by $h(\omega, x) := c R x + \xi(\omega)$, where $c \neq 0$ is a constant and ξ is a fixed arbitrary element of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$.

Note that the integral with respect to \overleftarrow{dB} is a “backward” Itô integral, while the integral with respect to dW is a standard “forward” Itô integral. We refer the reader to [3] for more details on such integrals, which are particular cases of the Itô-Skorohod stochastic integral.

A solution of (2.1) is a quintuple (y, Y, z, Z, k) of stochastic processes such that (y, Y, z, Z, k) belongs to \mathbb{M}^2 and satisfies the following FBDSDE:

$$\begin{cases} y_t = x_0 + \int_0^t b(s, y_s, Y_s, z_s, Z_s, k_s, v_s)ds + \int_0^t \sigma(s, y_s, Y_s, z_s, Z_s, k_s, v_s)dW_s \\ \quad + \int_0^t \int_{\Theta} \varphi(s, y_s, Y_s, z_s, Z_s, k_s, v_s, \rho) \tilde{N}(d\rho, ds) - \int_0^t z_s \overleftarrow{dB}_s, \\ Y(t) = h(y_T) + \int_t^T f(s, y_s, Y_s, z_s, Z_s, k_s, v_s)ds \\ \quad + \int_t^T g(s, y_s, Y_s, z_s, Z_s, k_s, v_s) \overleftarrow{dB}_s \\ \quad - \int_t^T Z_s dW_s - \int_t^T \int_{\Theta} k_s(\rho) \tilde{N}(d\rho, ds), \quad t \in [0, T]. \end{cases}$$

Define the cost functional by:

$$J(v) := \mathbb{E} \left[\int_0^T \ell(t, y_t, Y_t, z_t, Z_t, k_t, v_t)dt + \beta(y_T) + \gamma(Y_0) \right], \quad v \in \mathcal{U}_{ad}, \quad (2.2)$$

where

$$\begin{aligned}\beta &: \mathbb{R}^n \rightarrow \mathbb{R}, \\ \gamma &: \mathbb{R}^m \rightarrow \mathbb{R}, \\ \ell &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R}^{m \times d} \times L_\nu^2(\mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R},\end{aligned}$$

are measurable functions such that (2.2) is defined.

Now the control problem of system (2.1) is to minimize J over \mathcal{U}_{ad} . In this case we say that $u. \in \mathcal{U}_{ad}$ is an *optimal control* if

$$J(u.) = \inf_{v. \in \mathcal{U}_{ad}} J(v.). \quad (2.3)$$

Let us set the following notations:

$$\begin{aligned}\zeta &= (y, Y, z, Z, k) \in \mathbb{R}^{n+m+n \times l+m \times d} \times L_\nu^2(\mathbb{R}^m), \\ A(t, \zeta, v) &= (-R^*f, Rb, -R^*g, R\sigma, R\varphi)(t, \zeta, v), \\ \langle A, \zeta \rangle &= -\langle y, R^*f \rangle + \langle Y, Rb \rangle - \langle z, R^*g \rangle + \langle Z, R\sigma \rangle + \langle \langle k, R\varphi \rangle \rangle,\end{aligned}$$

where

$$\begin{aligned}R^*g &= (R^*g_1 \cdots R^*g_l), R\sigma = (R\sigma_1 \cdots R\sigma_d), \dots, \\ \langle \langle k, R\varphi \rangle \rangle(t, \zeta, v) &= \int_{\Theta} \langle k(\rho), R\varphi(t, \zeta, v, \rho) \rangle \nu(d\rho).\end{aligned}$$

The following assumptions will be our main assumptions in the paper. We shall mimic similar assumptions from the literature (e.g. [10]) for this purpose.

- (A1) $\forall \zeta = (y, Y, z, Z, k), \bar{\zeta} = (\bar{y}, \bar{Y}, \bar{z}, \bar{Z}, \bar{k}) \in \mathbb{R}^{n+m+n \times l+m \times d} \times L_\nu^2(\mathbb{R}^m), \forall t \in [0, T], \forall v \in U,$

$$\begin{aligned}\langle A(t, \zeta, v) - A(t, \bar{\zeta}, v), \zeta - \bar{\zeta} \rangle &\leq -\mu(|R(y - \bar{y})|^2 + |R(Y - \bar{Y})|^2 \\ &\quad + \|R^*(z - \bar{z})\|^2 + \|R^*(Z - \bar{Z})\|^2 + \|R^*(k - \bar{k})\|^2),\end{aligned}$$

and

$$c > 0,$$

or

- (A1)',

$$\begin{aligned}\langle A(t, \zeta, v) - A(t, \bar{\zeta}, v), \zeta - \bar{\zeta} \rangle &\geq \mu(|R(y - \bar{y})|^2 + |R(Y - \bar{Y})|^2 \\ &\quad + \|R^*(z - \bar{z})\|^2 + \|R^*(Z - \bar{Z})\|^2 + \|R^*(k - \bar{k})\|^2),\end{aligned}$$

and

$$c < 0,$$

where μ is a positive constant.

- (A2) For each $\zeta \in \mathbb{R}^{n+m+n \times l+m \times d} \times L^2_{\nu}(\mathbb{R}^m)$, $A(\cdot, \zeta)$ is a predictable process and $A(\cdot, 0) \in \mathbb{M}^2$.
- (A3)

$$\left\{ \begin{array}{l} (i) \text{ The mappings } f, b, g, \sigma, \varphi, \ell \text{ are continuously differentiable with respect to } (y, Y, z, Z, k) \text{ and } \beta \text{ are continuously differentiable with respect to } y, \\ \text{and } \gamma \text{ is continuously differentiable with respect to } Y, \\ (ii) \text{ the derivatives of } f, b, g, \sigma, \varphi \text{ with respect to the above arguments are} \\ \text{bounded,} \\ (iii) \text{ the derivatives of } \ell \text{ are bounded by } C(1 + |y| + |Y| + \|z\| + \|Z\| + \|k\|), \\ (iv) \text{ the derivatives of } \beta \text{ and } \gamma \text{ are bounded by } C(1 + |y|) \text{ and } C(1 + |Y|) \\ \text{respectively,} \end{array} \right.$$

for some constant $C > 0$.

Remark 2.1 *The condition $c > 0$ in (A1) guarantees the following monotonicity condition of the mapping h :*

$$\langle h(y) - h(\bar{y}), R(y - \bar{y}) \rangle \geq 0, \quad \forall y, \bar{y} \in \mathbb{R}^n.$$

The same thing happens also for $c < 0$ in (A1)'.

The following theorem is concerned with the existence and uniqueness of the solution of (2.1).

Theorem 2.2 *For any given admissible control v and under the assumptions (A1)–(A3) (or (A1)', (A2), (A3)), then (2.1) has a unique solution.*

Our assumptions in this theorem satisfy the assumptions of the corresponding result in [11], so the proof of this theorem can be gleaned from [11].

3 Adjoint equations and the maximum principle

Let u be an arbitrary element of \mathcal{U}_{ad} and (y, Y, z, Z, k) be the corresponding solution of (2.1). Suppose that (A1)–(A3) hold. We want here to introduce first the adjoint equations of the FBDSDE (2.1) and then present our main result of the maximum principle for optimal control of our system with jumps (2.1). To this end, let us begin by defining the Hamiltonian H from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times$

$\mathbb{R}^{n \times l} \times \mathbb{R}^{m \times d} \times L^2_\nu(\mathbb{R}^m) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times l} \times \mathbb{R}^{m \times d} \times L^2_\nu(\mathbb{R}^m)$ to \mathbb{R} by the formula:

$$\begin{aligned} H(t, y, Y, z, Z, k, v, p, P, q, Q, V) := & \langle p, f(t, y, Y, z, Z, k, v) \rangle \\ & - \langle q, b(t, y, Y, z, Z, k, v) \rangle + \langle P, g(t, y, Y, z, Z, k, v) \rangle \\ & - \langle Q, \sigma(t, y, Y, z, Z, k, v) \rangle - \ell(t, y, Y, z, Z, k, v) \\ & - \int_{\Theta} \langle V(\hat{\rho}), \varphi(t, y, Y, z, Z, k, v, \hat{\rho}) \rangle \nu(d\hat{\rho}). \end{aligned} \quad (3.1)$$

Now the adjoint equations of the FBDSDE with jumps (2.1) are

$$\begin{cases} dp_t = H_Y dt + H_Z dW_t - P_t \overleftarrow{dB}_t + \int_{\Theta} H_k \tilde{N}(d\rho, dt), \\ dq_t = H_y dt + H_z \overleftarrow{dB}_t + Q_t dW_t + \int_{\Theta} V_t(\rho) \tilde{N}(d\rho, dt), \\ p_0 = -\gamma_Y(Y_0), q_T = -c R p_T + \beta_y(y_T), \end{cases} \quad (3.2)$$

where H_y is the gradient $\nabla_y H(t, y, Y, z, Z, k, v, p, P, q, Q, V), \dots$ etc. Let us say some thing more about this system (3.2).

Theorem 3.1 *Under (A1)–(A3) there exists a unique solution (p, P, q, Q, V) of the adjoint equations (3.2) (in \mathbb{M}^2).*

Proof. This system (3.2) can be rewritten as in the following system:

$$\begin{cases} dp_t = (f_Y^* p_t - b_Y^* q_t + g_Y^* P_t - \sigma_Y^* Q_t - \int_{\Theta} \varphi_Y^* V_t(\hat{\rho}) \nu(d\hat{\rho}) - \ell_Y) dt \\ \quad + (f_Z^* p_t - b_Z^* q_t + g_Z^* P_t - \sigma_Z^* Q_t - \int_{\Theta} \varphi_Z^* V_t(\hat{\rho}) \nu(d\hat{\rho}) - \ell_Z) dW_t - P_t \overleftarrow{dB}_t \\ \quad + \int_{\Theta} (f_k^* p_t - b_k^* q_t + g_k^* P_t - \sigma_k^* Q_t - \int_{\Theta} \varphi_k^* V_t(\hat{\rho}) \nu(d\hat{\rho}) - \ell_k) \tilde{N}(d\rho, dt), \\ dq_t = (f_y^* p_t - b_y^* q_t + g_y^* P_t - \sigma_y^* Q_t - \int_{\Theta} \varphi_y^* V_t(\hat{\rho}) \nu(d\hat{\rho}) - \ell_y) dt \\ \quad + (f_z^* p_t - b_z^* q_t + g_z^* P_t - \sigma_z^* Q_t - \int_{\Theta} \varphi_z^* V_t(\hat{\rho}) \nu(d\hat{\rho}) - \ell_z) \overleftarrow{dB}_t + Q_t dW_t \\ \quad + \int_{\Theta} V_t(\rho) \tilde{N}(d\rho, dt), \\ p_0 = -\gamma_Y(Y_0), q_T = -c R p_T + \beta_y(y_T), \end{cases}$$

which is a linear FBDSDE with jumps. Here f_y is the gradient $\nabla_y f(t, y, Y, z, Z, k, v), \dots$ etc.

Thanks to assumptions (A1)–(A3) the result follows from Theorem 2.2. ■

Now our main theorem is the following.

Theorem 3.2 *Assume that (A1)–(A3) hold. Given $u. \in \mathcal{U}_{ad}$, let (y, Y, z, Z, k) and (p, P, q, Q, V) be the corresponding solutions of the FBDSDEs (2.1) and (3.2) respectively. Suppose that the following assumptions hold:*

- (i) β and γ are convex,
- (ii) for all $t \in [0, T]$, \mathbb{P} -a.s., the function $H(t, \cdot, \cdot, \cdot, \cdot, \cdot, p, P, q, Q, V)$ is concave,
- (iii) we have

$$\begin{aligned} & H(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) \\ &= \max_{v \in U} H(t, y_t, Y_t, z_t, Z_t, k_t, v, p_t, P_t, q_t, Q_t, V_t), \end{aligned} \quad (3.3)$$

for a.e. \mathbb{P} -a.s.

Then (y, Y, z, Z, k, u) is an optimal solution of the control problem (2.1)–(2.3).

The proof of this theorem will be established in Section 4. Now to illustrate this theorem let us present an example.

Example 3.3 Let $(\Theta, \mathcal{B}(\Theta)) = ([0, 1], \mathcal{B}([0, 1]))$. Let $\tilde{N}(d\rho, dt)$ be a compensated Poisson random measure, where $(t, \rho) \in [0, 1] \times [0, 1]$. Recall that $\mathbb{E}[\tilde{N}(d\rho, dt)^2] = \nu(d\rho)dt$ is a finite Borel measure such that $\int_{[0,1]} \rho^2 \nu(d\rho) < \infty$. Let the controls domain be $U = [-1, 1]$. Consider the following stochastic control system:

$$\begin{cases} dy_t = (1+t)v_t dt + (-z_t + Z_t + k_t + v_t)dW_t - z_t \overleftarrow{dB}_t - v_t \int_{[0,1]} \rho \tilde{N}(d\rho, dt), \\ dY_t = -(t-4)v_t dt - \frac{3}{2}(z_t + Z_t + k_t + v_t) \overleftarrow{dB}_t + Z_t dW_t \\ \quad \quad \quad + \int_{[0,1]} k_t(\rho) \tilde{N}(d\rho, dt), \\ y_0 = Y_1 = x \in \mathbb{R}, \quad t \in (0, 1), \end{cases} \quad (3.4)$$

where W, B are Brownian motions in \mathbb{R} , and W, B and \tilde{N} are mutually independent. Consider also a cost functional given for $v \in \mathcal{U}_{ad}$ by

$$J(v) = \frac{1}{2} \mathbb{E} \left[\int_0^1 (y_t^2 + Y_t^2 + z_t^2 + Z_t^2 + k_t^2 + v_t^2) dt \right] + \frac{1}{2} \mathbb{E}[y_1^2] + \frac{1}{2} \mathbb{E}[Y_0^2]. \quad (3.5)$$

We define the value function by

$$J(u^*) = \inf_{v \in \mathcal{U}_{ad}} J(v). \quad (3.6)$$

This system (3.4) can be related to the one in (2.1) by setting the following

mappings:

$$\begin{aligned}
\beta(y_t) &= \gamma(y_t) = \frac{1}{2}y_t^2, \\
h(y_t) &= y_t, \text{ i.e. } c = 1, \xi = 0, \\
\varphi(t, y_t, Y_t, z_t, Z_t, k_t, v_t, \rho) &= -v_t\rho, \\
b(t, y_t, Y_t, z_t, Z_t, k_t, v_t) &= (1+t)v_t, \\
f(t, y_t, Y_t, z_t, Z_t, k_t, v_t) &= (t-4)v_t, \\
g(t, y_t, Y_t, z_t, Z_t, k_t, v_t) &= \frac{3}{2}(z_t + Z_t + k_t + v_t), \\
\sigma(t, y_t, Y_t, z_t, Z_t, k_t, v_t) &= (-z_t + Z_t + k_t + v_t), \\
\ell(t, y_t, Y_t, z_t, Z_t, k_t, v_t) &= \frac{1}{2}(y_t^2 + Y_t^2 + z_t^2 + Z_t^2 + k_t^2 + v_t^2).
\end{aligned}$$

One can see easily that assumptions (A1), (A3) (and of course (A2)) are fulfilled for this system. Thus Theorem 2.2 guarantees the existence and uniqueness of the solution of (3.4). In fact (3.4) is a fully coupled linear FBDSDE with jumps.

Now letting $u \equiv 0$, we find from the construction of FBDSDEs with jumps (as for instance in [11]) that the corresponding solution $(y_t, Y_t, z_t, Z_t, k_t)$ of (3.4) equals $(x, x, 0, 0, 0)$, for all $t \in [0, 1]$.

Next notice that the adjoint equations of (3.4) are

$$\begin{cases} dp_t = -Y_t dt + (\frac{3}{2}P_t - Q_t - Z_t)dW_t - P_t \overleftarrow{dB}_t \\ \quad \quad \quad + \int_{[0,1]} (\frac{3}{2}P_t - Q_t - k_t) \tilde{N}(d\rho, dt), \\ dq_t = -y_t dt + (\frac{3}{2}P_t + Q_t - z_t) \overleftarrow{dB}_t + Q_t dW_t + \int_{[0,1]} V_t(\rho) \tilde{N}(d\rho, dt), \\ p_0 = -x, q_1 = -cp_1 + \beta_y(y_1), \quad t \in (0, 1), \end{cases} \quad (3.7)$$

if we recall that

$$\begin{aligned}
p_0 &= -\gamma_Y(Y_0) = -\gamma_Y(x) = -x, \\
p_t &= p_0 - \int_0^t Y_t dt = p_0 - x \int_0^t dt = -x - xt = -x(1+t),
\end{aligned}$$

Here we used the fact that p_0 is deterministic. Hence p_1, q_1 are deterministic since:

$$\begin{aligned}
p_1 &= -2x, \\
q_1 &= -cp_1 + \beta_y(y_1) = -p_1 + \beta_y(x) = 3x.
\end{aligned}$$

It follows that

$$q_t = q_1 + \int_t^1 y_t dt = 3x + x(1-t) = x(4-t).$$

In particular, $(p_t, q_t, P_t, Q_t, V_t) \equiv (-x(1+t), x(4-t), 0, 0, 0)$ is the unique solution of (3.7). These facts show that the Hamiltonian attains an explicit formula:

$$\begin{aligned}
H(t, y_t, Y_t, z_t, Z_t, k_t, v, p_t, P_t, q_t, Q_t, V_t) &= p_t(t-4)v - q_t(1+t)v \\
&\quad - \frac{3}{2}P_t(z_t + Z_t + k_t + v) - Q_t(-z_t + Z_t + k_t + v) \\
&\quad - \int_{[0,1]} v \widehat{\rho} V_t(\widehat{\rho}) \nu(d\widehat{\rho}) - \frac{1}{2}(y_t^2 + Y_t^2 + z_t^2 + Z_t^2 + k_t^2 + v^2) \\
&= -x(1+t)(t-4)v - x(1+t)(4-t)v - \frac{1}{2}v^2 - \frac{1}{2}x^2 - \frac{1}{2}x^2 \\
&= -\frac{1}{2}v^2 - x^2, \quad v \in U.
\end{aligned}$$

Hence

$$\begin{aligned}
H(t, y_t, Y_t, z_t, Z_t, k_t, v, p_t, P_t, q_t, Q_t, V_t) - H(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) \\
= -\frac{1}{2}v^2 - x^2 + \frac{1}{2}u_t^2 + x^2 = -\frac{1}{2}v^2 \leq 0, \quad \forall v \in U, \text{ a.e } t, \mathbb{P} - a.s.
\end{aligned}$$

As a result, condition (iii) of Theorem 3.2 holds here for $u. = 0$. Furthermore, all other conditions of Theorem 3.2 can be verified easily. Consequently,

$$(y., Y., z., Z., k., u.) \equiv (x, x, 0, 0, 0, 0)$$

is an optimal solution of the control problem (3.4)–(3.6).

For more applications of the theory of fully coupled FBDSDEs particularly in providing a probabilistic formula for the solution of a quasilinear SPDIE we refer the reader to [11, P. 15].

4 Proofs

In this section we shall establish the proof of Theorem 3.2. Let us recall first the following lemma.

Lemma 4.1 (Integration by parts) *Let $(\alpha, \widehat{\alpha}) \in [\mathcal{S}^2(0, T; \mathbb{R}^n)]^2$, $(\beta, \widehat{\beta}) \in [\mathcal{M}^2(0, T; \mathbb{R}^n)]^2$, $(\gamma, \widehat{\gamma}) \in [\mathcal{M}^2(0, T; \mathbb{R}^{n \times k})]^2$, $(\delta, \widehat{\delta}) \in [\mathcal{M}^2(0, T; \mathbb{R}^{n \times d})]^2$, and $(K, \widehat{K}) \in [\mathcal{N}_\eta^2(0, T; \mathbb{R}^m)]^2$. Assume that*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s \overleftarrow{dB}_s + \int_0^t \delta_s dW_s + \int_0^t \int_{\Theta} K_{s-}(\rho) \tilde{N}(d\rho, ds),$$

and

$$\hat{\alpha}_t = \hat{\alpha}_0 + \int_0^t \hat{\beta}_s ds + \int_0^t \hat{\gamma}_s d\overleftarrow{B}_s + \int_0^t \hat{\delta}_s dW_s + \int_0^t \int_{\Theta} \hat{K}_{s-}(\rho) \tilde{N}(d\rho, ds),$$

for $t \in [0, T]$. Then

$$\langle \alpha_T, \hat{\alpha}_T \rangle = \langle \alpha_0, \hat{\alpha}_0 \rangle + \int_0^T \langle \alpha_t, d\hat{\alpha}_t \rangle + \int_0^T \langle d\alpha_t, \hat{\alpha}_t \rangle + \int_0^T d\langle \alpha, \hat{\alpha} \rangle_t.$$

$$\begin{aligned} \mathbb{E}[\langle \alpha_T, \hat{\alpha}_T \rangle] &= \mathbb{E}[\langle \alpha_0, \hat{\alpha}_0 \rangle] + \mathbb{E}\left[\int_0^T \langle \alpha_t, d\hat{\alpha}_t \rangle\right] + \mathbb{E}\left[\int_0^T \langle d\alpha_t, \hat{\alpha}_t \rangle\right] \\ &\quad - \mathbb{E}\left[\int_0^T \langle \gamma_t, \hat{\gamma}_t \rangle dt\right] + \mathbb{E}\left[\int_0^T \langle \delta_t, \hat{\delta}_t \rangle dt\right] + \mathbb{E}\left[\int_0^T \int_{\Theta} \langle K_t, \hat{K}_t \rangle \nu(d\rho) dt\right]. \end{aligned}$$

This lemma can be deduced directly from Itô's formula with jumps (see e.g. [13]).

We now prove Theorem 3.2. We start with two lemmas.

Lemma 4.2 *Under (A1)–(A4) we have*

$$\begin{aligned} J(v.) - J(u.) &\geq \mathbb{E}[\langle q_T, y_T^v - y_T \rangle] + \mathbb{E}[\langle c R p_T, y_T^v - y_T \rangle] - \mathbb{E}[\langle p_0, Y_0^v - Y_0 \rangle] \\ &\quad + \mathbb{E}\left[\int_0^T (\ell(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - \ell(t, y_t, Y_t, z_t, Z_t, k_t, u_t)) dt\right]. \end{aligned} \quad (4.1)$$

Proof. Let $v.$ be an arbitrary element of \mathcal{U}_{ad} . Denote by $(y^v, Y^v, z^v, Z^v, k^v)$ the corresponding solution of (2.1). We have

$$\begin{aligned} J(v.) - J(u.) &= \mathbb{E}[\beta(y_T^v) - \beta(y_T)] + \mathbb{E}[\gamma(Y_0^v) - \gamma(Y_0)] \\ &\quad + \mathbb{E}\left[\int_0^T (\ell(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - \ell(t, y_t, Y_t, z_t, Z_t, k_t, u_t)) dt\right]. \end{aligned}$$

Since β and γ are convex, we obtain

$$\begin{aligned} \beta(y_T^v) - \beta(y_T) &\geq \langle \beta_y(y_T), y_T^v - y_T \rangle, \\ \gamma(Y_0^v) - \gamma(Y_0) &\geq \langle \gamma_Y(Y_0), Y_0^v - Y_0 \rangle, \end{aligned}$$

which imply that

$$\begin{aligned} J(v.) - J(u.) &\geq \mathbb{E}[\langle \beta_y(y_T), y_T^v - y_T \rangle] + \mathbb{E}[\langle \gamma_Y(Y_0), Y_0^v - Y_0 \rangle] \\ &\quad + \mathbb{E}\left[\int_0^T (\ell(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - \ell(t, y_t, Y_t, z_t, Z_t, k_t, u_t)) dt\right]. \end{aligned}$$

From the adjoint equation (3.1) and system (2.1), we have

$$p_0 = -\gamma_Y(Y_0), q_T = -c R p_T + \beta_y(y_T).$$

Thus (4.1) holds. ■

The following lemma contains duality relations between (2.1) and (3.2) (see the equivalent equations in the proof of Theorem 3.1).

Lemma 4.3 *Suppose that assumptions of Theorem 3.2 (in particular (A1)–(A3)) hold. Then*

$$\begin{aligned} & -\mathbb{E}[\langle p_0, Y_0^{v\cdot} - Y_0 \rangle] = -\mathbb{E}[\langle p_T, Y_T^{v\cdot} - Y_T \rangle] \\ & -\mathbb{E}\left[\int_0^T \langle p_t, f(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) - f(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \langle H_Y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Y_t^{v\cdot} - Y_t \rangle dt\right] \\ & -\mathbb{E}\left[\int_0^T \langle P_t, g(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) - g(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \langle H_Z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Z_t^{v\cdot} - Z_t \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \int_{\Theta} \langle H_k(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), \right. \\ & \quad \left. k_t^{v\cdot}(\rho) - k_t(\rho) \rangle \nu(d\rho) dt\right], \quad (4.2) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\langle q_T, y_T^{v\cdot} - y_T \rangle] &= \mathbb{E}\left[\int_0^T \langle q_t, b(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) \right. \\ & \quad \left. - b(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \langle H_y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), y_t^{v\cdot} - y_t \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \langle H_z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), z_t^{v\cdot} - z_t \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \langle Q_t, \sigma(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) - \sigma(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt\right] \\ & +\mathbb{E}\left[\int_0^T \int_{\Theta} \langle V_t(\rho), \varphi(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t, \rho) \right. \\ & \quad \left. - \varphi(t, y_t, Y_t, z_t, Z_t, k_t, u_t, \rho) \rangle \nu(d\rho) dt\right]. \quad (4.3) \end{aligned}$$

Proof. Applying Integration by parts (Lemma 4.1) to $\langle p_t, Y_t^{v\cdot} - Y_t \rangle$ gives

$$\begin{aligned}
\langle p_T, Y_T^{v\cdot} - Y_T \rangle &= \langle p_0, Y_0^{v\cdot} - Y_0 \rangle \\
&- \int_0^T \langle p_t, f(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) - f(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt \\
&- \int_0^T \langle p_t, (g(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) \\
&\quad - g(t, y_t, Y_t, z_t, Z_t, k_t, u_t)) \overleftarrow{dB}_t \rangle \\
&+ \int_0^T \int_{\Theta} \langle p_t, (k_t^{v\cdot}(\rho) - k_t(\rho)) \rangle \tilde{N}(d\rho, dt) \\
&+ \int_0^T \langle p_t, (Z_t^{v\cdot} - Z_t) dW_t \rangle - \int_0^T \langle Y_t^{v\cdot} - Y_t, P_t \overleftarrow{dB}_t \rangle \\
&+ \int_0^T \langle H_Y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Y_t^{v\cdot} - Y_t \rangle dt \\
&+ \int_0^T \langle Y_t^{v\cdot} - Y_t, H_Z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) dW_t \rangle \\
&+ \int_0^T \int_{\Theta} \langle Y_t^{v\cdot} - Y_t, H_k(t, y_t, Y_t, z_t, Z_t, k_t, u_t, \\
&\quad p_t, P_t, q_t, Q_t, V_t) \rangle \tilde{N}(d\rho, dt) \\
&- \int_0^T \langle P_t, g(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t) - g(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt \\
&+ \int_0^T \langle H_Z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Z_t^{v\cdot} - Z_t \rangle dt \\
&+ \int_0^T \int_{\Theta} \langle H_k(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), \\
&\quad k_t^{v\cdot}(\rho) - k_t(\rho) \rangle \nu(d\rho) dt.
\end{aligned}$$

Now by taking the expectation to the above equality, we obtain (4.2).

Similarly

$$\begin{aligned}
\langle q_T, y_T^v - y_T \rangle &= \int_0^T \langle q_t, b(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) \\
&\quad - b(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt \\
&+ \int_0^T \langle q_t, (\sigma(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) \\
&\quad - \sigma(t, y_t, Y_t, z_t, Z_t, k_t, u_t)) dW_t \rangle \\
&+ \int_0^T \int_{\Theta} \langle y_t^v - y_t, V_t(\rho) \rangle \tilde{N}(d\rho, dt) \\
&- \int_0^T \langle q_t, (z_t^v - z_t) d\overleftarrow{B}_t \rangle + \int_0^T \langle y_t^v - y_t, Q_t dW_t \rangle \\
&+ \int_0^T \int_{\Theta} \langle q_t, (\varphi(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t, \rho) \\
&\quad - \varphi(t, y_t, Y_t, z_t, Z_t, k_t, u_t, \rho)) \rangle \tilde{N}(d\rho, dt) \\
&+ \int_0^T \langle y_t^v - y_t, H_z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) d\overleftarrow{B}_t \rangle \\
&+ \int_0^T \langle H_y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), y_t^v - y_t \rangle dt \\
&+ \int_0^T \langle H_z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), z_t^v - z_t \rangle dt \\
&+ \int_0^T \langle Q_t, \sigma(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) \\
&\quad - \sigma(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle dt \\
&+ \int_0^T \int_{\Theta} \langle V_t(\rho), \varphi(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t, \rho) \\
&\quad - \varphi(t, y_t, Y_t, z_t, Z_t, k_t, u_t, \rho) \rangle \nu(d\rho) dt.
\end{aligned}$$

By taking the expectation to this equality (4.3) holds. ■

The remaining is devoted to completing the proof of Theorem 3.2.

Proof of Theorem 3.2. Observe first from (3.1) that

$$\begin{aligned}
& \ell(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - \ell(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \\
&= - \left(H(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t, p_t, P_t, q_t, Q_t, V_t) \right. \\
&\quad \left. - H(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) \right) \\
&\quad + \langle p_t, f(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - f(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle \\
&\quad - \langle q_t, b(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - b(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle \\
&\quad + \langle P_t, g(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - g(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle \\
&\quad - \langle Q_t, \sigma(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t) - \sigma(t, y_t, Y_t, z_t, Z_t, k_t, u_t) \rangle \\
&\quad - \int_{\Theta} \langle V_t(\rho), \varphi(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t, \rho) \\
&\quad \quad - \varphi(t, y_t, Y_t, z_t, Z_t, k_t, u_t, \rho) \rangle \nu(d\rho). \tag{4.4}
\end{aligned}$$

Next apply Lemma 4.2, Lemma 4.3 and (4.4) to find that

$$\begin{aligned}
& J(v) - J(u) \\
&\geq \mathbb{E} \left[\int_0^T \langle H_y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), y_t^v - y_t \rangle dt \right] \\
&\quad + \mathbb{E} \left[\int_0^T \langle H_Y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Y_t^v - Y_t \rangle dt \right] \\
&\quad + \mathbb{E} \left[\int_0^T \langle H_z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), z_t^v - z_t \rangle dt \right] \\
&\quad + \mathbb{E} \left[\int_0^T \langle H_Z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Z_t^v - Z_t \rangle dt \right] \\
&\quad + \mathbb{E} \left[\int_0^T \int_{\Theta} \langle H_k(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), k_t^v(\rho) - k_t(\rho) \rangle \nu(d\rho) dt \right] \\
&\quad - \mathbb{E} \left[\int_0^T \left(H(t, y_t^v, Y_t^v, z_t^v, Z_t^v, k_t^v, v_t, p_t, P_t, q_t, Q_t, V_t) \right. \right. \\
&\quad \quad \left. \left. - H(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) \right) dt \right]. \tag{4.5}
\end{aligned}$$

On the other hand, from the concavity condition (ii) of the mapping

$$(y, Y, z, Z, k, v) \mapsto H(t, y, Y, z, Z, k, v, p, P, q, Q, V)$$

it follows that

$$\begin{aligned}
& H(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t, p_t, P_t, q_t, Q_t, V_t) \\
& - H(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t) \\
& \leq \langle H_y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), y_t^{v\cdot} - y_t \rangle \\
& + \langle H_Y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Y_t^{v\cdot} - Y_t \rangle \\
& + \langle H_z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), z_t^{v\cdot} - z_t \rangle \\
& + \langle H_Z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Z_t^{v\cdot} - Z_t \rangle \\
& + \int_{\Theta} \langle H_k(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), k_t^{v\cdot}(\rho) - k_t(\rho) \rangle \nu(d\rho) \\
& + \langle H_v(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), v_t - u_t \rangle.
\end{aligned}$$

In particular,

$$\begin{aligned}
& - \langle H_v(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), v_t - u_t \rangle \\
& \leq \langle H_y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), y_t^{v\cdot} - y_t \rangle \\
& + \langle H_Y(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Y_t^{v\cdot} - Y_t \rangle \\
& + \langle H_z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), z_t^{v\cdot} - z_t \rangle \\
& + \langle H_Z(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), Z_t^{v\cdot} - Z_t \rangle \\
& + \int_{\Theta} \langle H_k(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), k_t^{v\cdot}(\rho) - k_t(\rho) \rangle \nu(d\rho) \\
& - [H(t, y_t^{v\cdot}, Y_t^{v\cdot}, z_t^{v\cdot}, Z_t^{v\cdot}, k_t^{v\cdot}, v_t, p_t, P_t, q_t, Q_t, V_t) \\
& - H(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t)].
\end{aligned}$$

Now by applying this latter result in (4.5) we obtain

$$J(v) - J(u) \geq -\mathbb{E}\left[\int_0^T \langle H_v(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), v_t - u_t \rangle dt\right]. \quad (4.6)$$

On the other hand, the maximum condition (iii) yields

$$\langle H_v(t, y_t, Y_t, z_t, Z_t, k_t, u_t, p_t, P_t, q_t, Q_t, V_t), v_t - u_t \rangle \leq 0.$$

Hence (4.6) becomes

$$J(v) - J(u) \geq 0.$$

Since u is an arbitrary element of \mathcal{U}_{ad} , this inequality completes the proof if we recall (2.3). ■

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